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BUBBLY FLOW—I

A SIMPLIFIED MODEL

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Abstract—A simple set of equations for bubbly flow through a vertical tube is rigorously derived by means of a scale analysis from the realistic equations posed by Pauchon. It is shown that under steady flow conditions the void fraction will relax from its value at the inlet towards an asymptotic value within only a short distance of the inlet. The relationship between the inlet void fraction and the imposed pressure drop is discussed, and a simple expression is derived for the equilibrium void fraction.

Key Words: bubbly flow, gas, liquid, two-phase flow, mathematical model, void fraction

1. INTRODUCTION

In the field of two-phase flow, there is a need for models that are not only realistic, but are also simple in form. Simplicity is an asset in large computer codes such as are used to analyse cooling systems for nuclear reactors. Simple models are also useful when sophisticated computing facilities are not readily available. In this paper we shall simplify a recent model for bubbly flow through a vertical tube. The simplification will be done in a rigorous manner. Asymptotic methods will be used to determine which terms in the equations are small and may therefore be neglected. This validity of the resulting set of equations will then be tested by using them to obtain a simple expression for the void fraction in steady bubbly flow, and comparing the predictions of this expression with experimental data.

The model that we shall be simplifying is the one recently presented by Pauchon & Banerjee (1988). It is a one-dimensional model for adiabatic bubbly flow through a vertical tube, the equations having been derived from the full three-dimensional equations by performing instantaneous space-averaging. The continuity and momentum equations for the gas and liquid phases are as follows:

$$(\epsilon_G \rho_G)_t + (\epsilon_G \rho_G u_G)_z = 0, \tag{1a}$$

$$(\epsilon_L \rho_L)_t + (\epsilon_L \rho_L u_L)_z = 0, \tag{1b}$$

$$(\epsilon_G \rho_G u_G)_t + (D_G \epsilon_G \rho_G u_G^2)_z + \epsilon_G p_{Gz} + \epsilon_G \rho_L C_{VM}[(u_{Gt} + u_G u_{Gz}) - (u_{Lt} + u_L u_{Lz})]$$
$$= -\epsilon_G \rho_G g - \frac{2}{D} f_i \epsilon_G \rho_L (u_G - u_L)|u_G - u_L| \tag{1c}$$

and

$$(\epsilon_L \rho_L u_L)_t + (D_L \epsilon_L \rho_L u_L^2)_z + \epsilon_L p_{Gz} - \epsilon_G \rho_L C_{VM}[(u_{Gt} + u_G u_{Gz}) - (u_{Lt} + u_L u_{Lz})]$$
$$+ (H\epsilon_L \rho_L (u_G - u_L)^2)_z = -\frac{2}{D} f_w \rho_L u_L^2 - \epsilon_L \rho_L g + \frac{2}{D} f_i \epsilon_G \rho_L (u_G - u_L)|u_G - u_L|. \tag{1d}$$

The subscripts "t" and "z" indicate differentiation with respect to time and distance along the tube, while the subscript K indicates that the quantity pertains to the Kth phase ($K = L$ for liquid or G for gas). The symbols ϵ_K, ρ_K and u_K represent, respectively, the proportion of the cross-section

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occupied by the phase K, the density of phase K and the velocity of the Kth phase. The volume fractions, ϵ_K, satisfy

$$\epsilon_L = 1 - \epsilon_G, \tag{2}$$

and the two phases are assumed incompressible, so that the densities, ρ_K, are constant.

Other symbols in the equations are: D_L and D_G, the profile coefficients for liquid and gas; p_G, the mean pressure associated with the gas inside the bubbles within some cross-section of the tube; D, the diameter of the tube; C_{VM}, the coefficients of added mass; H, which represents the combined effects of Reynolds stresses and the distortion of liquid streamlines by the bubbles; f_w, the friction factor for the shear stress exerted at the tube wall; and f_i, the friction factor for the interfacial shear stress. Pauchon & Banerjee (1988) used the following values for C_{VM} and H, which are appropriate for single, isolated, spherical bubbles:

$$C_{VM} = \tfrac{1}{2} \tag{3}$$

and

$$H = \tfrac{1}{4} + \tfrac{1}{5}\epsilon_G. \tag{4}$$

(They also derived expressions for C_{VM} and H which sought to account for interactions between bubbles; however, when used to derive speeds for voidage waves, these other expressions did not appear to give better results than those obtained in [3] and [4], and so we shall use the latter here.)

For the interfacial friction factor Pauchon & Banerjee (1988) used

$$f_i = \frac{3}{8}\frac{D}{D_b}c_D, \tag{5}$$

where D_b is a representative bubble diameter and c_D is the drag coefficient, which they defined to be

$$c_D = \frac{24(1 + 0.1\,\mathrm{Re}_{2p})}{\mathrm{Re}_{2p}}, \tag{6}$$

where Re_{2p} is the two-phase Reynolds number:

$$\mathrm{Re}_{2p} = \frac{D_b\rho_L(u_G - u_L)\epsilon_L}{\eta_L}; \tag{7}$$

η_L being the liquid viscosity. For the data that we shall be analysing in this paper it is appropriate to use a different expression for the drag coefficient. We shall use

$$c_D = \frac{\kappa D_b}{2}\frac{[1 + 17.67(\epsilon_L^{3/2})^{6/7}]^2}{(18.67\epsilon_L^{3/2})^2}, \tag{8}$$

where

$$\kappa = \frac{4}{3}\sqrt{\frac{g(\rho_L - \rho_G)}{\sigma}}, \tag{9}$$

g is the acceleration due to gravity and σ is the surface tension. This expression for c_D was derived by Ishii & Zuber (1979) for flows in Newton's regime, i.e. when

$$\mathrm{Re}_{2p} \gtrsim 40. \tag{10}$$

Ishii & Zuber (1979) argued that [8] for c_D could also be applied to bubbles which are distorted, which occurs when the flow satisfies

$$0.11\left(\frac{1 + \Psi}{\Psi^{8/3}}\right) < \eta_L[g(\rho_L - \rho_G)]^{1/4}(\rho_L\sigma^{3/2})^{-1/2},$$

where

$$\Psi = 0.055\{[1 + 0.01(D_b^*)^3]^{4/7} - 1\}^{3/4}$$

and

$$D_b^* = D_b[\rho_L(\rho_L - \rho_G)g]^{1/3}\eta_L^{-2/3}.$$

For the wall friction factor we use the Blasius relation (Mercadier 1981, p. 81):

$$f_w = \frac{0.314\,\mathrm{Re}_L^{-1/4}}{4}, \qquad [11]$$

where

$$\mathrm{Re}_L = \frac{\rho_L u_L \epsilon_L D}{\eta_L}. \qquad [12]$$

This completes the specification of the model we shall be considering in this paper. For such a model describing cocurrent flow through a vertical tube, we would expect to prescribe boundary conditions at the base of the tube. For the model to be deemed realistic it should therefore be hyperbolic. Pauchon & Banerjee (1988) have demonstrated that if the differential terms in the equations are formulated as outlined above, the model is indeed hyperbolic under flow conditions corresponding to bubbly flow. This model is therefore a reasonable starting point for our analysis.

The boundary conditions we expect to prescribe with this model are ϵ_G at the inlet, together with the two mass fluxes

$$G_G \triangleq \rho_G \epsilon_G u_G$$

and

$$G_L \triangleq \rho_L \epsilon_L u_L.$$

These give initial conditions for ϵ_G, u_G and u_L. The system of equations can then in principle be solved by eliminating the pressure gradient between the two momentum equations. Once the state of the system (ϵ_G, u_G, u_L) has been obtained, then the pressure drop along the tube could be derived by integrating an expression for the pressure gradient from [1c] or [1d] along the tube. A prescribed pressure drop could be satisfied by suitable adjustment of the value of ϵ_G at the inlet. The prescription of boundary conditions in this manner appears straightforward; however, it contains many subtleties. In section 2 we shall demonstrate that the boundary condition on ϵ_G can be dropped when considering an asymptotic approximation for the flow away from the inlet. Possible repercussions of such neglect are discussed in section 3.

2. DERIVATION OF THE SCALED EQUATIONS

In this section we shall perform a scale analysis to determine the relative sizes of the various terms in [1a–d]. This will guide us in the next section, when we decide which terms may be neglected.

We define dimensionless variables (as indicated by the superscript "+") in the following manner:

$$z = Lz^+, \qquad t = Tt^+, \qquad \epsilon_L = e_L \epsilon_L^+, \qquad \epsilon_G = e_G \epsilon_G^+,$$
$$u_L = U_L u_L^+, \quad u_G = U_G u_G^+, \quad p = p_{sys} + Pp^+, \qquad [13]$$

where p_{sys} is a representative pressure of the system such as the pressure at the flow outlet. The scales L, T, e_L, e_G, U_L, U_G and P should be chosen so that the dimensionless variables are each of order one.

It is possible to specify scales in an *ad hoc* manner based on experimental observations of the values of various equations under given flow conditions. However, here we shall *derive* suitable expressions for the scales from the equations themselves. This requires only that we make some preliminary assumptions as to which are the dominant effects in the process described by [1a–d].

We start by considering the momentum equation for the liquid. In the liquid, the pressure drop along the tube will be due mainly to the hydrostatic head. If we define the scale L to be the length of the pipe, then balancing the pressure gradient and gravitational terms in [1d] gives a suitable scale for P:

$$P = \rho_L g L. \qquad [14]$$

Next, we consider [1c] in order to derive an appropriate value for e_G. The reason for doing this will become clear in the next section. There we will discover that, although we might prescribe the void fraction at the inlet (ϵ_{Gin}) as a boundary condition equivalent to imposing a pressure drop along the tube, in fact the value of the void fraction may change significantly from its boundary value within only a short distance from the inlet. Thus, ϵ_{Gin} may not be a suitable value for the scale e_{G}. In order to derive an appropriate value for e_{G}, we will balance two terms in [1c]; the force on the bubbles due to the pressure gradient and the opposing interfacial friction force, as given by [5] and [8]. Because the void fraction is small in bubbly flow, it is possible to expand [8] as a power series in ($e_{\mathrm{G}}\epsilon_{\mathrm{G}}^{+}$). The algebra is recorded in detail in the thesis by Seward (1988); here we merely note that by doing this, and by using [14] for P, we can derive an approximate expression for e_{G}:

$$e_{\mathrm{G}} \simeq \left\{\frac{\rho_{\mathrm{G}} G_{\mathrm{Lin}}}{\rho_{\mathrm{L}} G_{\mathrm{Gin}}} - 1.132 + \sqrt{2.69 + 2.264\frac{\rho_{\mathrm{G}} G_{\mathrm{Lin}}}{\rho_{\mathrm{L}} G_{\mathrm{Gin}}} + 2.66\frac{g}{\kappa}\left[\frac{\rho_{\mathrm{G}}}{G_{\mathrm{Gin}}}\right]^2}\right\}^{-1}, \qquad [15]$$

where κ is defined by [9], and $G_{K\mathrm{in}}$ are the values of the mass fluxes at the flow inlet.

If a simpler expression had been chosen for the interfacial friction, such as [6], then it would not have been necessary to assume that the void fraction was small in order to derive an expression for e_{G}. The assumption of small void fraction would also not have been necessary, had a value for e_{G} merely been derived numerically from a balance between a frictional term using [8] and the pressure relation. Thus, it is possible to derive values for e_{G} directly from [1c] without making assumptions as to the magnitude of e_{G} or the relative sizes of other parameters such as U_{L} and U_{G}. Given [15], values can be defined for U_{G} and U_{L} which are consistent with the equations for the conservation of mass, assuming that the mass fluxes at the inlet, $G_{K\mathrm{in}}$, are prescribed:

$$U_{\mathrm{G}} = \frac{G_{\mathrm{Gin}}}{\rho_{\mathrm{G}} e_{\mathrm{G}}}, \qquad [16]$$

$$e_{\mathrm{L}} = 1 - e_{\mathrm{G}} \qquad [17]$$

and

$$U_{\mathrm{L}} = \frac{G_{\mathrm{Lin}}}{\rho_{\mathrm{L}} e_{\mathrm{L}}}. \qquad [18]$$

For values G_{Gin} and G_{Lin} appropriate to bubbly flow, it is found that, in general,

$$U_{\mathrm{G}} > U_{\mathrm{L}}.$$

An appropriate time scale for an analysis of the system is then the slowest response time of the system. Thus, we define

$$T = \frac{L}{U_{\mathrm{L}}}. \qquad [19]$$

The scales [14]–[19] form a consistent set for [1a–d]. They are valid in general for bubbly flow and Seward (1988) used these scales to non-dimensionalize her equations. However, the purpose of this paper is to derive *simple* forms for the equations. It is therefore reasonable to use [14]–[19] together with a particular range of flow conditions to determine the relative sizes of these scales, and hence arrive in a reasoned manner at a second set of scales. Using data from Mercadier's (1981) thesis (which we shall quote below), it can be shown that

$$\frac{U_{\mathrm{L}}}{U_{\mathrm{G}}} \simeq 1$$

and

$$e_{\mathrm{L}} \simeq 1.$$

We deduce that alternative scales for e_{L} and e_{G} are, respectively,

$$e_{\mathrm{G}} = c_1 = \frac{\rho_{\mathrm{L}} G_{\mathrm{Gin}}}{\rho_{\mathrm{G}} G_{\mathrm{Lin}}} \qquad [20]$$

and

$$e_{\mathrm{L}} = 1. \qquad [21]$$

Using these new definitions for e_G and e_L, and leaving the other scales defined as before, a particularly simple form of [1a–d] results.

If [1a–d] is rewritten in terms of the dimensionless variables and their associated scales we obtain:

$$\epsilon^+_{G_{t^+}} + (\epsilon^+_G u^+_G)^+_{z^+} = 0, \qquad [22a]$$

$$-c_1\epsilon^+_{G_{t^+}} + ((1 - c_1\epsilon^+_G)u^+_L)_{z^+} = 0, \qquad [22b]$$

$$\delta_1\{\delta_2[(\epsilon^+_G u^+_G)_{t^+} + (D_G\epsilon^+_G u^{+2}_G)_{z^+}] + \tfrac{1}{2}C^+_{VM}\epsilon^+_G[(u^+_{G_{t^+}} + u^+_G u^+_{G_{z^+}}) - (u^+_{L_{t^+}} u^+_L u^+_{L_{z^+}})]\}$$
$$+ \epsilon^+_G p^+_{G_{z^+}} = -\delta_2\epsilon^+_G - \epsilon^+_G(u^+_G - u^+_L)^2 f^+_i \qquad [22c]$$

and

$$\delta_1\{[((1 - c_1\epsilon^+_G)u^+_L)_{t^+} + (D_L(1 - c_1\epsilon^+_G)u^{+2}_L)_{z^+}] + \tfrac{1}{4}[H^+(1 - c_1\epsilon^+_G)(u^+_G - u^+_L)^2]_{z^+}$$
$$- (\tfrac{1}{2})c_1 C^+_{VM}\epsilon^+_G[(u^+_{G_{t^+}} + u^+_G u^+_{G_{z^+}}) - (u^+_{L_{t^+}} + u^+_L u^+_{L_{z^+}})]\}$$
$$+ (1 - c_1\epsilon^+_G)p^+_{G_{z^+}} = -(1 - c_1\epsilon^+_G) - \delta_3 u^{+2}_L + c_1\epsilon^+_G(u^+_G - u^+_L)^2 f^+_i. \qquad [22d]$$

In these equations, the dimensionless parameters are defined as follows:

$$H^+ = 4H, \qquad C^+_{VM} = 2C_{VM},$$

$$c_1 = \frac{G_{Gin}\,\rho_L}{G_{Lin}\,\rho_G}, \qquad \delta_1 = \frac{G^2_{Lin}}{g\rho^2_L L},$$

$$\delta_2 = \frac{\rho_G}{\rho_L}, \qquad \delta_3 = \frac{G^2_{Lin} f_w}{g\rho^2_L D},$$

$$f^+_i = \frac{2G^2_{Lin} f_i}{g\rho^2_L D} = \frac{[1 + 2.266(c_1\epsilon^+_G) + 3.785(c_1\epsilon_G)^2 + O(c_1\epsilon_G)^3]}{s^2} \qquad [23]$$

and

$$s = \left(\frac{8g}{3\kappa}\right)^{1/2}\frac{\rho_L}{G_{Lin}}, \qquad \kappa = \frac{4}{3}\left[\frac{g(\rho_L - \rho_G)}{\sigma}\right]^{1/2}. \qquad [24]$$

The magnitude of the dimensionless parameters can be determined using data obtained by Mercadier (1981). He performed experiments on bubbly flow in a cylindrical annulus, which was formed from two concentric tubes of outer and inner radii 0.016 and 0.035 m, respectively. He used air and water in the test section at 20°C and atmospheric pressure. Typical values of the densities, viscosities and surface tension would have been

$$\rho_L = 1000\ \mathrm{kg\,m^{-3}}, \quad \eta_L = 1.1 \times 10^{-3}\ \mathrm{kg\,m^{-1}\,s^{-1}},$$
$$\rho_G = 1.62\ \mathrm{kg\,m^{-3}}, \quad \eta_G = 1.76 \times 10^{-5}\ \mathrm{kg\,m^{-1}\,s^{-1}} \qquad [25]$$

and

$$\sigma = 0.073\ \mathrm{N\,m^{-2}}.$$

His data for the mass fluxes, G_{Kin}, and bubble diameters, D_b, lay mainly between the two extremes:

$$\text{(i)}\ G_{Lin} = 91.25\ \mathrm{kg\,m^{-2}\,s^{-1}}, \quad G_{Gin} = 0.0148\ \mathrm{kg\,m^{-2}\,s^{-1}} \quad (D_b = 0.0030\ \mathrm{m})$$

and

$$\text{(ii)}\ G_{Lin} = 912.5\ \mathrm{kg\,m^{-2}\,s^{-1}}, \quad G_{Gin} = 0.2957\ \mathrm{kg\,m^{-2}\,s^{-1}} \quad (D_b = 0.0060\ \mathrm{m}).$$

We deduce that for Mercadier's (1981) data:

$$c_1 = O(10^{-1}), \quad \delta_1 = O(10^{-2}), \quad \delta_2 = O(10^{-3}), \quad \delta_3 = O(10^{-3}). \qquad [26]$$

This information tells us the sizes of the various terms in [22], when these equations are applied to flow conditions similar to that of Mercadier. However, we note that under other flow conditions the magnitudes of these scales may be slightly different. If the pipe is sufficiently long, then δ_1 may be of order 10^{-3}, or even less. Also, while the value of e_G given by [15] would most likely be of order 10^{-1} under flow conditions near the transition to slug flow, the revised scale, c_1, may be close to 1.

3. SIMPLIFICATION OF THE EQUATIONS FOR BUBBLY FLOW

In this section we shall obtain a simple set of equations for bubbly flow by determining which terms in [22a–d] may be neglected. Our criteria for neglecting a term are that the term should be at most of order 10^{-2} and, moreover, should correspond to a regular perturbation. A regular perturbation of a system of equations is a term which does not affect the order of the equations, i.e. which is not a highest derivative. Thus (dropping the superscript "+" for dimensionless variables in this section and in section 5), algebraic terms such as $\delta_2 \epsilon_G^+$ in [22c] and $\delta_3 u_L^2$ in [22d] are regular terms. The derivatives

$$\delta_2[(\epsilon_G u_G)_t + (D_G \epsilon_G u_G^2)_z] \qquad [27]$$

in [22c] are also regular since they are not required in order to satisfy boundary conditions at the inlet. This is because in [22c] the term

$$\tfrac{1}{2} C_{VM}(u_{Gt} + u_G u_{Gz})$$

contains derivatives of u_G which are far more significant than those in [27]. Likewise, other terms in [22c] are more significant than [27] for the application of a boundary condition on ϵ_G.

A singular perturbation is one which does alter the order of the equations. Removal of singularly perturbed terms generally means that not all boundary conditions can be satisfied. For example, if the derivative terms in the continuity equations were neglected, it would not be possible to satisfy boundary conditions on the mass fluxes at the inlet. These terms should therefore not be neglected in any simplification of [22a–d].

In the rest of this section we shall be concerned with the remaining terms of $O(10^{-2})$ in [22a–d]. That is,

$$\delta_1 \{\tfrac{1}{2} C_{VM} \epsilon_G [(u_{Gt} + u_G u_{Gz}) - (u_{Lt} + u_L u_{Lz})]\} \qquad [28]$$

in [22c] and

$$\delta_1 \{[((1 - c_1 \epsilon_G) u_L)_t + (D_L (1 - c_1 \epsilon_G) u_L^2)_z] + \tfrac{1}{4}[H(1 - c_1 \epsilon_G)(u_G - u_L)^2]_z$$
$$- \tfrac{1}{2} C_{VM} \epsilon_G [(u_{Gt} + u_G u_{Gz}) - (u_{Lt} + u_L u_{Lz})]\} \qquad [29]$$

in [22d]. We wish to establish if they constitute a regular or a singular perturbation and whether or not they may be neglected.

For clarity we shall restrict ourselves to considering the behaviour of the system in the steady-state. Under such conditions, the fluxes of liquid and gas entering the base of the vertical tube are prescribed as constant; [22a] and [22b] therefore give

$$u_G = \frac{1}{\epsilon_G} \qquad [30]$$

and

$$u_L = \frac{1}{(1 - c_1 \epsilon_G)}. \qquad [31]$$

After omitting the small regular terms from [22c] and [22d], the pressure gradient may be eliminated between these two equations to produce an equation containing the variables ϵ_G, u_G and u_L.

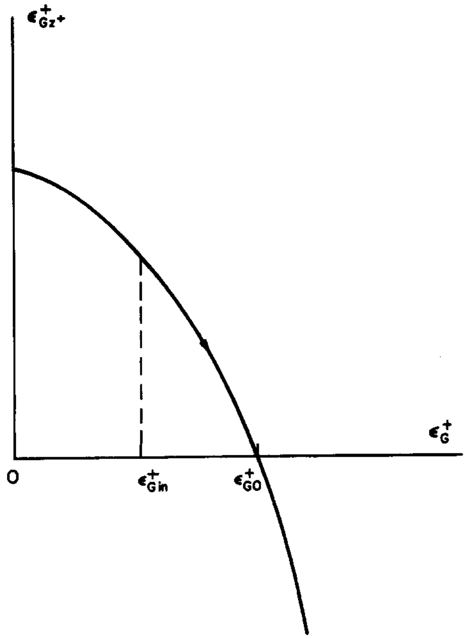

Figure 1. The rate of change of the dimensionless void fraction with distance along the tube (ϵ^+_{Gz}); ϵ^+_G will relax to its approximate asymptotic value ϵ^+_{G0}.

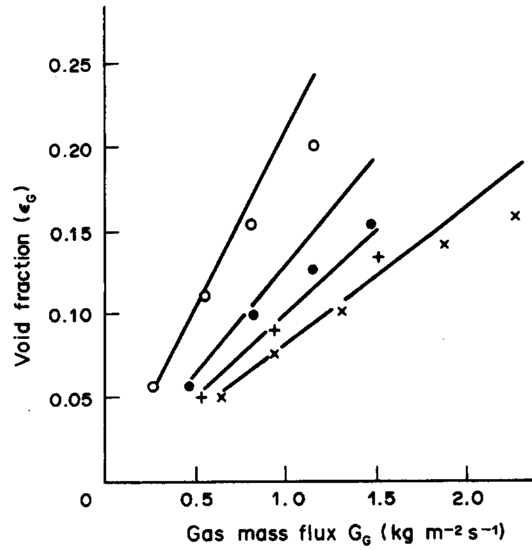

Figure 2. Predictions of the voidage fraction using [36], compared with the data of Micaelli (1982): G_L(kg m^{-2} s^{-1}) = 3000 (○), 5000 (●), 6500 (+), 8000 (×).

Equations [30] and [31] may then be used to eliminate u_G and u_L from this equation, and so we obtain the following equation for ϵ_G:

$$\delta_1\{D_L c_1(1-c_1\epsilon_G)\epsilon_G^3+\tfrac{1}{4}[-2(1-c_1\epsilon_G)^3+(2+c_1)\epsilon_G(1-c_1\epsilon_G)^3+c_1(1-c_1\epsilon_G)\epsilon_G^3]$$
$$+\tfrac{1}{2}[(1-c_1\epsilon_G)^3+c_1\epsilon_G^3]\}\epsilon_{Gz}=-(1-c_1\epsilon_G)\epsilon_G^3+[(1-c_1\epsilon_G)-\epsilon_G]^2\epsilon_G(1-c_1\epsilon_G)\left(\frac{1}{s^2}\right)$$
$$\times[1+2.266(c_1\epsilon_G)+3.785(c_1\epsilon_G)^2+O(c_1\epsilon_G)^3]. \qquad [32]$$

Equation [32] is a first-order differential equation describing how the value of ϵ_G varies with distance along the tube. Since δ_1 is small, we might expect to be able to ignore the derivative term. However, this is a *singular* perturbation, and we may expect this term to be significant near the inlet. In order to investigate the behaviour near the inlet we shall neglect all regular terms of order c_1. (Later we will see from figure 2 that such an assumption leads to reasonable estimates of the void fraction, implying that the assumption is in practice valid.) Neglecting terms of $O(c_1)$, we obtain

$$\delta_1\epsilon_{Gz}=\frac{(1-\epsilon_G)^2}{s^2}-\epsilon_G^2. \qquad [33]$$

If we assume that the gas is rising faster than the liquid at the inlet, then

$$\epsilon_{Gin}<1,$$

and [33] can be represented by figure 1. Clearly, the dimensionless void fraction will relax to the approximate asymptotic value

$$\epsilon_{G0}=\frac{1}{(1+s)} \qquad [34]$$

within a short distance [of $O(\delta_1)$, dimensionlessly] of the inlet. Thus, if we are concerned only with the flow at a reasonable distance from the inlet we may neglect [28] and [29].

An exceedingly simple set of equations now results if, in addition, we neglect all the regular terms of $O(10^{-2})$ which have been indicated in this section:

$$\epsilon_{Gt} + (\epsilon_G u_G)_z = 0, \qquad [35a]$$

$$-c_1\epsilon_{Gt} + ((1 - c_1\epsilon_G)u_L)_z = 0 \qquad [35b]$$

and

$$(1 - c_1\epsilon_G) = -p_z = f_i(u_G - u_L)|u_G - u_L|, \qquad [35c]$$

where

$$f_i = \left(\frac{1}{s^2}\right)[1 + 2.266(c_1\epsilon_G) + 3.785(c_1\epsilon_G)^2 + O(c_1\epsilon_G)^3].$$

These equations will be used in Part II of this paper (Lisseter & Fowler 1992, this issue, pp. 205–215) to study the propagation of void fraction waves through bubbly flow.

4. PREDICTION OF THE VOID FRACTION IN THE STEADY-STATE

In figure 1 we saw that under steady flow conditions the dimensionless void fraction relaxed to the approximate value,

$$\epsilon^+_{G0} = \frac{1}{(1+s)},$$

with increasing distance from the flow inlet. The corresponding dimensional value for the void fraction would be

$$c_1\epsilon^+_{G0} = \frac{c_1}{(1+s)}. \qquad [36]$$

In figure 2 we compare predictions of the void fraction made using [36] with data from the experiments by Micaelli (1982). Micaelli performed his experiments on bubbly flow through a vertical tube of square cross-section (0.02 × 0.02 m). Air and water were used in the test-section at room temperature and a pressure of 6 bar. Under some flow conditions Micaelli found that the *mean* velocity of the gas was less than that of the liquid. This was due to the way in which the void and velocity distributions varied over the cross-section of the tube. However, since we assumed in deriving [34] that the gas was travelling faster than the liquid, we shall omit such data from our comparison.

From figure 2 we see that the accuracy of the predictions worsens as the gas flux increases relative to the liquid flux; i.e. as c_1 increases. However, we can obtain a higher order (in c_1) approximation to the asymptotic value of the void fraction with distance along the tube. We derive this from [32] by setting $\epsilon_{Gz} = 0$, and equating terms through $O(c_1)$. Our revised expression for the dimensional void fraction is thus:

$$c_1\epsilon^+_G = c_1\epsilon^+_{G0} + c_1^2\epsilon^+_{G1} + O(c_1^3), \qquad [37]$$

where

$$\epsilon^+_{G0} = \frac{1}{(1+s)} \qquad [38]$$

and

$$\epsilon^+_{G1} = \epsilon^{+2}_{G0}[2.633(1 - \epsilon^+_{G0}) - 1]. \qquad [39]$$

Predictions made using this expression are shown in figure 3, and we see that the accuracy is markedly improved at the higher values of the gas flux, and the correct trends are displayed.

Mercadier (1981) used a wider range of liquid mass fluxes than Micaelli, and so the prediction of his data (for which the flow conditions were described in section 2) presents a more stringent test of our expression for the void fraction. Results from [37]–[39] are shown in figure 4. They are good.

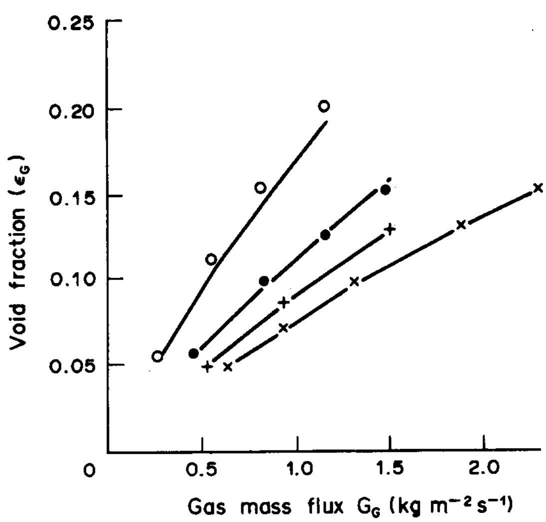

Figure 3. Predictions of the voidage fraction using [37], compared with the data of Micaelli (1982). Symbols as in figure 2.

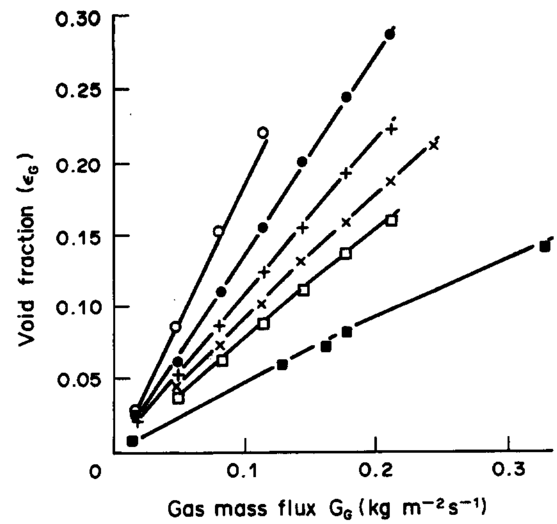

Figure 4. Predictions of the voidage fraction using [37], compared with the data of Mercadier (1981): G_L(kg m^{-2} s^{-1}) = 91.25 (○), 182.5 (●), 273.8 (+), 365.0 (×), 456.3 (□), 912.5 (■).

From these results we can conclude that [37]–[39] give good predictions of the asymptotic value attained by the void fraction within a short distance of the tube inlet.

5. THE PRESCRIPTION OF BOUNDARY CONDITIONS FOR NUMERICAL ROUTINES

Since the void fraction may vary rapidly within a short boundary layer next to the flow inlet, numerical routines for the solution of the differential equations should be constructed with care. To illustrate this point we will now consider the relationship, under steady flow conditions, between the value of $\epsilon_{\rm Gin}$ which is prescribed as a boundary condition at the flow inlet and the (dimensionless) pressure drop along the tube, Δp.

From [22c,d] we have

$$-p_{\mathrm{G}z} = \epsilon_{\mathrm{L}} + \delta_1\left[\frac{D_{\mathrm{L}}}{\epsilon_{\mathrm{L}}} + \frac{1}{4}\epsilon_{\mathrm{L}}\left(\frac{1}{\epsilon_{\mathrm{G}}} - \frac{1}{\epsilon_{\mathrm{L}}}\right)^2\right]_z,$$

so

$$\Delta p = \epsilon_{\mathrm{L}0} + \delta_1\left[\frac{D_{\mathrm{L}}}{\epsilon_{\mathrm{L}}} + \frac{1}{4}\epsilon_{\mathrm{L}}\left(\frac{1}{\epsilon_{\mathrm{G}}} - \frac{1}{\epsilon_{\mathrm{L}}}\right)^2\right]_{\epsilon_{\mathrm{Gin}}}^{\epsilon_{\mathrm{G}0}} + O\left(\frac{\delta_1 c_1(\epsilon_{\mathrm{G}0} - \epsilon_{\mathrm{Gin}})}{2}\right), \qquad [40]$$

where

$$\epsilon_{\mathrm{L}} = (1 - c_1\epsilon_{\mathrm{G}}), \quad \epsilon_{\mathrm{G}0} = \frac{1}{(1+s)}, \quad \epsilon_{\mathrm{L}0} = (1 - c_1\epsilon_{\mathrm{G}0}).$$

As we mentioned in section 2, it is Δp rather than $\epsilon_{\rm Gin}$ which may be considered to be the physically prescribed boundary condition; so that given some (dimensionless) pressure drop Δp, we might wish to use [40] to determine a suitable value for $\epsilon_{\rm Gin}$ for use as an equivalent boundary condition. However, we note that in [40] δ_1 is $O(10^{-2})$, and so $\epsilon_{\rm Gin}$ is very sensitive to the value of Δp (and in numerical routines, to the mesh size in the inlet region.) If the pressure drop (or inlet void fraction) boundary condition is instead ignored, then the work of section 4 shows that satisfactory approximations to the flow can be made by ignoring inertial terms in the momentum equations. However, such an approximation will lead to a computed pressure drop, $\Delta p \sim \epsilon_{\rm L0}$ which will be *inaccurate*. In fact, [40] implies that if Δp is prescribed and greater than $\epsilon_{\rm L0}$, then the inlet region will act as a constriction to the liquid flow. On the other hand, if one tried to impose a pressure drop which is $<\epsilon_{L0}$, then no physical value of ϵ_{Gin} can satisfy [40] and we might conclude that the tube would not contain bubbly flow, but rather slug flow.

6. CONCLUSION

In this paper we have shown how, when given a set of realistic equations for bubbly flow, scales can be constructed for various quantities in a logical manner. Using these scales, the equations were then non-dimensionalized. It was indicated how to identify those terms which could reasonably be neglected from these equations, and a set of simple, time-dependent equations was thereby derived ([35a–c]). Under steady flow conditions, the equations could be simplified still further to give an expression ([36]–[39]) for the approximate value of the void fraction away from the inlet region of the flow. Predictions of this expression were found to compare well with experimental data from the theses by Mercadier (1981) and Micaelli (1982). The final section of this paper considered the relationship between the imposed pressure drop and the value of the void fraction. It was concluded that numerical routines for calculating the pressure drop using the full set of equations [1a–d] and a full set of boundary conditions should take care to treat the inlet region correctly.

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